Some Connections Between (Sub)Critical Branching Mechanisms and Bernstein Functions

Jean Bertoin, Bernard Roynette, Marc Yor

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Jean Bertoin[(1)], Bernard Roynette[(2)], and Marc Yor[(1)]

(1) *Laboratoire de Probabilités et Modèles Aléatoires and Institut universitaire de France, Université Pierre et Marie Curie, 175, rue du Chevaleret, F-75013 Paris, France.*

(2) *Institut Elie Cartan, Campus Scientifique, BP 239, Vandoeuvre-lès-Nancy Cedex F-54056, France*

Abstract

We describe some connections, via composition, between two functional spaces: the space of (sub)critical branching mechanisms and the space of Bernstein functions. The functions $\mathbf{e}_\alpha : x \to x^\alpha$ where $x \geq 0$ and $0 < \alpha \leq 1/2$, and in particular the critical parameter $\alpha = 1/2$, play a distinguished role.

1 Introduction

This note is a prolongation of [8] where the following remarkable property of the function $\mathbf{e}_\alpha : x \to x^\alpha$ was pointed at for $\alpha = 1/2$: if Ψ is a (sub)critical branching mechanism, then $\Psi \circ \mathbf{e}_{1/2}$ is a Bernstein function (see the next section for the definition of these notions). In the present work, we first show that this property extends to every $\alpha \in]0, 1/2]$. Then we characterize the class of so-called internal functions, i.e. that of Bernstein functions Φ such that the compound function $\Psi \circ \Phi$ is again a Bernstein function for every (sub)critical branching mechanism Ψ. In the final section, we gather classical results on transformations of completely monotone functions, Bernstein functions and (sub)critical branching mechanisms which are used in our analysis.

2 Some functional spaces

2.1 Completely monotone functions

For every Radon measure μ on $[0, \infty[$, we associate the function $\mathcal{L}_\mu :]0, \infty[\to [0, \infty]$ defined by

$$\mathcal{L}_\mu(q) := \int_{]0,\infty[} \mathrm{e}^{-qx} \mu(dx) \,, \tag{1}$$

i.e. $\mathcal{L}_\mu$ is the Laplace transform of μ. We denote by

$$\mathbf{CM} := \{\mathcal{L}_\mu : \mathcal{L}_\mu(q) < \infty \text{ for all } q > 0\} \ , \tag{2}$$

which is an algebraic convex cone (i.e. a convex cone which is further stable under inner product). The celebrated theorem of Bernstein (see for instance Theorem 3.8.13 in [6]) identifies **CM** with the space of completely monotone functions, i.e. functions $f :]0,\infty[\to [0,\infty[$ of class $\mathcal{C}^\infty$ such that for every integer $n \geq 1$, the n-th derivative $f^{(n)}$ of f has the same sign as $(-1)^n$. Recall from monotone convergence that $\mathcal{L}_\mu$ has a (possibly infinite) limit at $0+$ which coincides with the total mass of μ.

We shall focus on two natural sub-cones of **CM**:

$$\mathbf{B}_1 := \left\{ \mathcal{L}_\mu : \int_{]0,\infty[} (1 \wedge x^{-1}) \mu(dx) < \infty \right\} \tag{3}$$

We further denote by $\mathbf{B}_1^\downarrow$ the sub-space of functions in $\mathbf{B}_1$ which are the Laplace transforms of absolutely continuous measures with a decreasing density :

$$\mathbf{B}_1^\downarrow := \left\{ \mathcal{L}_\mu : \mu(dx) = g(x)dx, g \text{ decreasing and } \int_0^\infty (1 \wedge x^{-1}) g(x) dx < \infty \right\}. \tag{4}$$

Note that the density g then has limit 0 at infinity.

2.2 Bernstein functions

For every triple (a, b, Λ) with $a, b \geq 0$ and Λ a positive measure on $]0,\infty[$ such that

$$\int_{]0,\infty[} (x \wedge 1) \Lambda(dx) < \infty , \tag{5}$$

we associate the function $\Phi_{a,b,\Lambda} :]0,\infty[\to [0,\infty[$ defined by

$$\Phi_{a,b,\Lambda}(q) := a + bq + \int_{]0,\infty[} (1 - \mathrm{e}^{-qx}) \Lambda(dx) , \tag{6}$$

and call $\Phi_{a,b,\Lambda}$ the Bernstein function with characteristics (a, b, Λ). We denote the convex cone of Bernstein functions by

$$\mathbf{B}_2 := \{ \Phi_{a,b,\Lambda} : a, b \geq 0 \text{ and } \Lambda \text{ positive measure fulfilling (5)} \} . \tag{7}$$

It is well-known that $\mathbf{B}_2$ can be identified with the space of real-valued $\mathcal{C}^\infty$ functions $f :]0,\infty[\to [0,\infty[$ such that for every integer $n \geq 1$, the n-th derivative $f^{(n)}$ of f has the same sign as $(-1)^{n-1}$. See Definition 3.9.1 and Theorem 3.9.4 in [6].

Bernstein functions appear as Laplace exponents of subordinators, see e.g. Chapter 1 in [3], Chapter 6 in [9], or Section 3.9 in [6]. This means that $\Phi \in \mathbf{B}_2$ if and only if there exists an increasing process $\sigma = (\sigma_t, t \geq 0)$ with values in $[0,\infty]$ (∞ serves as absorbing state) with independent and stationary increments as long as $\sigma_t < \infty$, such that for every $t \geq 0$

$$\mathbb{E}(\exp(-q\sigma_t)) = \exp(-t\Phi(q)) , \qquad q > 0.$$

In this setting, a is known as the killing rate, b as the drift coefficient, and Λ as the Lévy measure.

We shall further denote by $\mathbf{B}_2^\downarrow$ the subspace of Bernstein functions for which the Lévy measure is absolutely continuous with a monotone decreasing density, viz.

$$\mathbf{B}_2^\downarrow := \left\{ \Phi_{a,b,\Lambda} : a, b \geq 0 \text{ and } \Lambda(dx) = g(x)dx, g \geq 0 \text{ decreasing and } \int_0^\infty (x \wedge 1) g(x) dx < \infty \right\}.$$

2.3 (Sub)critical branching mechanisms

For every triple (a,b,Π) with $a,b\geq 0$ and Π positive measure on $]0,\infty[$ such that

$$\int_{]0,\infty[}(x\wedge x^2)\Pi(dx)<\infty \tag{8}$$

we associate the function $\Psi_{a,b,\Pi}:]0,\infty[\to[0,\infty[$ defined by

$$\Psi_{a,b,\Pi}(q):=aq+bq^2+\int_{]0,\infty[}(\mathrm{e}^{-qx}-1+qx)\Pi(dx)\,, \tag{9}$$

and denote the convex cone of such functions by

$$\mathbf{B}_3:=\{\Psi_{a,b,\Pi}:a,b\geq 0\text{ and }\Pi\text{ a positive measure such that (8) holds}\} \tag{10}$$

Functions in $\mathbf{B}_3$ are convex increasing functions of class $\mathcal{C}^\infty$ that vanish at 0; they coincide with the class of branching mechanisms for (sub)critical continuous state branching processes, where (sub)critical means critical or sub-critical. See Le Gall [7] on page 132.

Alternatively, functions in the space $\mathbf{B}_3$ can also be viewed as Laplace exponents of Lévy processes with no positive jumps that do not drift to $-\infty$ (or, equivalently, with nonnegative mean). In this setting, a is the drift coefficient, $2b$ the Gaussian coefficient, and Π the image of the Lévy measure by the map $x\to -x$. See e.g. Chapter VII in [2].

3 Composition with $\mathbf{e}_\alpha$

Stable subordinators correspond to a remarkable one-parameter family of Bernstein functions denoted here by $(\mathbf{e}_\alpha, 0<\alpha<1)$, where

$$\mathbf{e}_\alpha(q):=q^\alpha\;=\;\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty(1-\mathrm{e}^{-qx})x^{-1-\alpha}dx\,,\qquad q>0\,.$$

Theorem 1 *The following assertions are equivalent:*

(i) $\alpha\in]0,1/2]$.

(ii) *For every* $\Psi\in\mathbf{B}_3$, $\Psi\circ\mathbf{e}_\alpha\in\mathbf{B}_2$.

The implication (ii) $\Rightarrow$ (i) is immediate. Indeed, $\Psi_{0,1,0}:q\to q^2$ belongs to $\mathbf{B}_3$, but $\mathbf{e}_{2\alpha}=\Psi_{0,1,0}\circ\mathbf{e}_\alpha$ is in $\mathbf{B}_2$ if and only if $2\alpha\leq 1$. However, the converse (i) $\Rightarrow$ (ii) is not straightforward and relies on the following technical lemma, which appears as Lemma VI.1.2 in [8]. Here, for the sake of completeness, we provide a proof.

Lemma 2 *For* $\alpha\in]0,1/2]$, *let* $\sigma^{(\alpha)}=(\sigma^{(\alpha)}_x, x\geq 0)$ *be a stable subordinator with index* α *with Laplace transform*

$$\mathbb{E}\left(\exp\left(-q\sigma^{(\alpha)}_x\right)\right)=\exp(-xq^\alpha)\,,\qquad x,q>0\,.$$

Denote by $p^{(\alpha)}(x,t)$ *the density of the law of* $\sigma^{(\alpha)}_x$. *Then for every* $,x,t>0$, *we have*

$$p^{(\alpha)}(x,t)\leq\frac{\alpha}{\Gamma(1-\alpha)}xt^{-(1+\alpha)}\,.$$

Remark : The bound in Lemma 2 is sharp, as it is well-known that for any $0 < \alpha < 1$ and each fixed $t > 0$

$$p^{(\alpha)}(x,t) \sim \frac{\alpha}{\Gamma(1-\alpha)} x t^{-(1+\alpha)}\,, \qquad x \to \infty.$$

More precisely, there is a series representation of $p^{(\alpha)}(x,t)$, see Formula (2.4.7) on page 90 in Zolotarev [10]:

$$p^{(\alpha)}(x,1) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{\Gamma(n+1))}\sin(\pi n\alpha)x^{-n\alpha-1}\,.$$

Using the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}\,,$$

this agrees of course with the above estimate. It is interesting to note that the second leading term in the expansion,

$$-\frac{\Gamma(2\alpha+1)}{2\pi}\sin(2\pi\alpha)x^{-2\alpha-1},$$

is negative for $\alpha < 1/2$, but positive for $\alpha > 1/2$. So the bound in Lemma 2 would fail for $\alpha > 1/2$.

Proof: In the case $\alpha = 1/2$, there is an explicit expression for the density

$$p^{(1/2)}(x,t) \;=\; \frac{x}{2\sqrt{\pi t^3}}\exp\left(-\frac{x^2}{4t}\right)\,,$$

from which the claim is obvious (recall that $\Gamma(1/2) = \sqrt{\pi}$).

In the case $\alpha < 1/2$, we start from the identity

$$\exp(-xq^\alpha) \;=\; \int_0^\infty \mathrm{e}^{-qt}p^{(\alpha)}(x,t)dt\,,$$

and take the derivative in the variable q to get

$$\alpha q^{\alpha-1}\exp(-xq^\alpha) \;=\; \int_0^\infty \mathrm{e}^{-qt}\frac{t}{x}p^{(\alpha)}(x,t)dt\,,$$

and then

$$\alpha q^{\alpha-1}\left(1-\exp(-xq^\alpha)\right) \;=\; \int_0^\infty \mathrm{e}^{-qt}\left(\frac{\alpha}{\Gamma(1-\alpha)}t^{-\alpha} - \frac{t}{x}p^{(\alpha)}(x,t)\right)dt\,.$$

Denote the left hand-side by $g(x,q)$, and take the derivative in the variable x. We obtain

$$\frac{\partial g(x,q)}{\partial x} \;=\; \alpha q^{2\alpha-1}\mathrm{e}^{-xq^\alpha} \;=\; \alpha q^{2\alpha-1}\int_0^\infty \mathrm{e}^{-qt}p^{(\alpha)}(x,t)dt\,.$$

On the other hand, since $1-2\alpha > 0$,

$$q^{2\alpha-1} \;=\; \frac{1}{\Gamma(1-2\alpha)}\int_0^\infty \mathrm{e}^{-qs}s^{-2\alpha}ds\,,$$

and hence

$$\frac{\partial g(x,q)}{\partial x} \;=\; \frac{\alpha}{\Gamma(1-2\alpha)}\int_0^\infty \frac{ds}{s^{2\alpha}}\int_0^\infty dt\mathrm{e}^{-q(s+t)}p^{(\alpha)}(x,t)\,.$$

The change of variables $u = t + s$ yields

$$\frac{\partial g(x,q)}{\partial x} = \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^\infty du \int_0^u \frac{ds}{s^{2\alpha}} \mathrm{e}^{-qu} p^{(\alpha)}(x, u-s)\,;$$

and since $g(0,t) = 0$, we finally obtain the identity

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-qt} \left(\frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x,t) \right) dt \\ = \ & \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^x dy \int_0^\infty du \int_0^u \frac{ds}{s^{2\alpha}} \mathrm{e}^{-qu} p^{(\alpha)}(x, u-s)\,. \end{aligned}$$

Inverting the Laplace transform, we conclude that

$$\frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x,t) = \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^x dy \int_0^t \frac{ds}{s^{2\alpha}} p^{(\alpha)}(x, t-s)\,,$$

which entails our claim. ■

We are now able to prove Theorem 1.

Proof: Let $\Psi_{a,b,\Pi} \in \mathbf{B}_3$. Since both $a\mathbf{e}_\alpha$ and $b\mathbf{e}_{2\alpha}$ are Bernstein functions, there is no loss of generality in assuming that $a = b = 0$. Set for $t > 0$

$$\nu_\alpha(t) := \frac{\alpha}{\Gamma(1-\alpha) t^{1+\alpha}} \int_0^\infty \Pi(dx) x \left(1 - \frac{\Gamma(1-\alpha) t^{1+\alpha}}{\alpha x} p^{(\alpha)}(x,t) \right).$$

It follows from Lemma 2 that $\nu_\alpha(t) \geq 0$. We have for every $q > 0$

$$\begin{aligned} &\int_0^\infty (1 - \mathrm{e}^{-qt}) \nu_\alpha(t) dt \\ = \ & \int_0^\infty \Pi(dx) x \int_0^\infty dt \left(\frac{\alpha(1-\mathrm{e}^{-qt})}{\Gamma(1-\alpha) t^{1+\alpha}} - \frac{p^{(\alpha)}(x,t)}{x} + \mathrm{e}^{-qt} \frac{p^{(\alpha)}(x,t)}{x} \right) \\ = \ & \int_0^\infty \Pi(dx) x \left(q^\alpha - \frac{1}{x} + \frac{\mathrm{e}^{-q^\alpha x}}{x} \right) \\ = \ & \Psi_{0,0,\Pi}(\mathbf{e}_\alpha(q))\,. \end{aligned}$$

As this quantity is finite for every $q > 0$, this shows that $\Psi_{0,0,\Pi} \circ \mathbf{e}_\alpha \in \mathbf{B}_2$. ■

Remark : The proof gives a stronger result than that stated in Theorem 1. Indeed, we specified the Lévy measure ν_α of $\Psi_{0,0,\Pi} \circ \mathbf{e}_\alpha$. Furthermore, in the case $\alpha = 1/2$, this expression shows that $\Psi_{0,0,\Pi} \circ \mathbf{e}_{1/2} \in \mathbf{B}_2^\downarrow$. It is interesting to combine this observation with the forthcoming Proposition 7 : for every $\Psi \in \mathbf{B}_3$, $\Psi \circ \mathbf{e}_{1/2} \in \mathbf{B}_2^\downarrow$, thus $\mathrm{Id} \times (\Psi \circ \mathbf{e}_{1/2}) : q \to q\Phi(\sqrt{q})$ is again in $\mathbf{B}_3$, and in turn $\mathbf{e}_{1/2} \times (\Psi \circ \mathbf{e}_{1/4}) \in \mathbf{B}_2^\downarrow$. More generally, we have by iteration that for every integer n

$$\mathbf{e}_{2-2^{1-n}} \times (\Psi \circ \mathbf{e}_{2^{-n}}) \in \mathbf{B}_3\,,$$

and

$$\mathbf{e}_{1-2^{-n}} \times (\Psi \circ \mathbf{e}_{2^{-n-1}}) \in \mathbf{B}_2^\downarrow\,.$$

4 Internal functions

It is well-known that the cone **CM** of completely monotone functions and the cone $\mathbf{B}_2$ of Bernstein functions are both stable by right composition with a Bernstein function; see Proposition 8 below. Theorem 1 incites us to consider also compositions of (sub)critical branching mechanisms and Bernstein functions; we make the following definition :

Definition 3 *A Bernstein function $\Phi \in \mathbf{B}_2$ is said* internal *if $\Psi \circ \Phi \in \mathbf{B}_2$ for every $\Psi \in \mathbf{B}_3$.*

Theorem 1 shows that the functions $\mathbf{e}_\alpha$ are internal if and only if $\alpha \in]0, 1/2]$. The critical parameter $\alpha = 1/2$ plays a distinguished role. Indeed, we could also prove Theorem 1 using the following alternative route. First, we check that $\mathbf{e}_{1/2}$ is internal (see [8]), and then we deduce by subordination that for every $\alpha < 1/2$ that $\Psi \circ \mathbf{e}_\alpha = \Psi \circ \mathbf{e}_{1/2} \circ \mathbf{e}_{2\alpha}$ is again a Bernstein function for every $\Psi \in \mathbf{B}_3$. Developing this argument, we easily arrive at the following characterization of internal functions :

Theorem 4 *Let $\Phi = \Phi_{a,b,\Lambda} \in \mathbf{B}_2$ be a Bernstein function. The following assertions are then equivalent:*

(i) Φ *is internal,*

(ii) $\Phi^2 \in \mathbf{B}_2$,

(iii) $b = 0$ *and there exists a subordinator $\sigma = (\sigma_t, t \geq 0)$ such that*

$$\Lambda(dx) \;=\; c \int_0^\infty t^{-3/2} \mathbb{P}(\sigma_t \in dx) dt\,.$$

Proof: (i) $\Rightarrow$ (ii) is obvious as $\Psi_{0,1,0} \circ \Phi = \Phi^2$.

(ii) $\Rightarrow$ (i). We know from Theorem 1 or [8] that for every $\Psi \in \mathbf{B}_3$, $\Psi \circ \mathbf{e}_{1/2} \in \mathbf{B}_2$. It follows by subordination that for every Bernstein function $\kappa \in \mathbf{B}_2$, $\Psi \circ \mathbf{e}_{1/2} \circ \kappa \in \mathbf{B}_2$. Take $\kappa = \Phi^2$, so $\mathbf{e}_{1/2} \circ \kappa = \Phi$, and hence Φ is internal.

(iii) $\Rightarrow$ (ii) Let κ denote the Bernstein function of σ. We have

$$\begin{aligned}
\Phi(q) &= a + \int_{]0,\infty[} (1 - \mathrm{e}^{-qx}) \Lambda(dx) \\
&= a + c \int_{]0,\infty[} \int_0^\infty dt (1 - \mathrm{e}^{-qx}) t^{-3/2} \mathbb{P}(\sigma_t \in dx) \\
&= a + c \int_0^\infty dt (1 - \mathrm{e}^{-t\kappa(q)}) t^{-3/2}\,.
\end{aligned}$$

The change of variables $t\kappa(q) = u$ yields

$$\Phi(q) \;=\; a + c' \sqrt{\kappa(q)}$$

and hence

$$\Phi^2(q) \;=\; a^2 + 2ac' \sqrt{\kappa(q)} + c'^2 \kappa(q)\,.$$

Since $\kappa^{1/2} = \mathbf{e}_{1/2} \circ \kappa$ is again a Bernstein function, we thus see that $\Phi^2 \in \mathbf{B}_2$.

(ii) $\Rightarrow$ (iii) Recall that the drift coefficient b of $\Phi_{a,b,\Lambda}$ is given by

$$\lim_{q\to\infty} \Phi_{a,b,\Lambda}(q)/q \;=\; b\,;$$

see e.g. page 7 in [3]. It follows immediately that $b = 0$ whenever $\kappa := \Phi^2_{a,b,\Lambda} \in \mathbf{B}_2$. Recall from Sato [9] on page 197-8 that if $\tau^{(1)}$ and $\tau^{(2)}$ are two independent subordinators with respective Bernstein functions $\Phi^{(1)}$ and $\Phi^{(2)}$, then the compound process $\tau^{(1)} \circ \tau^{(2)} := \tau^{(3)}$ is again a subordinator with Bernstein function $\Phi^{(3)} := \Phi^{(2)} \circ \Phi^{(1)}$; moreover its Lévy measure $\Lambda^{(3)}$ is given by

$$\Lambda^{(3)}(dx) = \int_0^\infty \mathbb{P}(\tau^{(1)}_t \in dx)\Lambda^{(2)}(dt)\,,$$

where $\Lambda^{(2)}$ denotes the Lévy measure of $\tau^{(2)}$. As $\Phi_{a,b,\Lambda} = \mathbf{e}_{1/2} \circ \kappa$, and the Lévy measure of $\mathbf{e}_{1/2}$ is $ct^{-3/2}dt$ with $c = 1/(2\sqrt{\pi})$, we deduce that

$$\Lambda(dx) \;=\; c\int_0^\infty \mathbb{P}(\sigma_t \in dx)t^{-3/2}dt\,.$$

The proof of Theorem 4 is now complete. ∎

It is noteworthy that if $\Phi_{a,b,\Lambda}$ is internal and $\Lambda \not\equiv 0$, then

$$\int_{]0,\infty} x\Lambda(dx) \;=\; \infty\,.$$

Indeed,

$$\int_{]0,\infty} x\Lambda(dx) \;=\; c\int_0^\infty \int_{]0,\infty[} x\mathbb{P}(\sigma_t \in dx)t^{-3/2}dt \;=\; c\int_0^\infty \mathbb{E}(\sigma_1)t^{-1/2}dt \;=\; \infty\,.$$

For instance, the Bernstein function $q \to \log(1+q)$ of the gamma subordinator is not internal.

Corollary 5 *For every $\Psi \in \mathbf{B}_3$, wewrite Φ for the inverse function of Ψ and then Φ' for its derivative. Then $1/\Phi'$ is internal.*

Proof: It is known (see Corollary 10 below) that $1/\Phi'$ is a Bernstein function; let us check that its square is also a Bernstein function.

We know that $\Psi'' \in \mathbf{B}_1$ (Proposition 6 below) and $\Phi \in \mathbf{B}_2$ (Proposition 9 below); we deduce from Proposition 8 that $\Psi'' \circ \Phi \in \mathbf{B}_1$. If we write $I(f) : x \to \int_0^x f(y)dy$ for every locally integrable function f, then again by Proposition 6, we get that $I(\Psi'' \circ \Phi)$ is a Bernstein function.

Now

$$\Psi'' \;=\; -\frac{\Phi'' \circ \Psi}{(\Phi' \circ \Psi)^3}\,,$$

so

$$\Psi'' \circ \Phi \;=\; -\frac{\Phi''}{(\Phi')^3}\,,$$

and we conclude that

$$\frac{1}{2(\Phi')^2} \;=\; I(\Psi'' \circ \Phi) \in \mathbf{B}_2\,.$$

∎

5 Some classical results and their consequences

For convenience, this section gathers some classical transformations involving $\mathbf{B}_j$, $j \in \{1,2,3\}$ and related subspaces, which have been used in the preceding section. We start by considering derivatives and indefinite integrals. The following statement is immediate.

Proposition 6 *Let $j = 2, 3$ and $f :]0,\infty[\to [0,\infty[$ be a $\mathcal{C}^\infty$-function with derivative f'. For $j = 3$, we further suppose that $\lim_{q\to 0} f(q) = 0$. There is the equivalence*

$$f \in \mathbf{B}_j \iff f' \in \mathbf{B}_{j-1}\,.$$

The next statement is easily checked using integration by parts.

Proposition 7 *Let $j = 2, 3$ and consider two functions $f, g :]0,\infty[\to [0,\infty[$ which are related by the identity $f(q) = qg(q)$. Then there is the equivalence*

$$f \in \mathbf{B}_j \text{ and } \lim_{q\to 0} f(q) = 0 \iff g \in \mathbf{B}^{\downarrow}_{j-1}\,.$$

Proposition 7 has well-known probabilistic interpretations. First, let σ be a subordinator with Bernstein function $f \in \mathbf{B}_2$ with unit mean, viz. $\mathbb{E}(\sigma_1) = 1$, which is equivalent to $f'(0+) = 1$. Then the completely monotone function $g(q) := f(q)/q$ is the Laplace transform of a probability measure on $\mathbb{R}_+$. The latter appears in the renewal theorem for subordinators (see e.g. [4]); in particular it describes the weak limit of the so-called age process $A(t) = t - g_t$ as $t \to \infty$, where $g_t := \sup\{\sigma_s : \sigma_s < t\}$. Second, let X be a Lévy process with no positive jumps and Laplace exponent $f \in \mathbf{B}_3$. The Lévy process reflected at its infimum, $X_t - \inf_{0\le s\le t} X_s$, is Markovian; and if τ denotes its inverse local time at 0, then $\sigma = -X \circ \tau$ is a subordinator called the descending ladder-height process. The Bernstein function of the latter is then given by $g(q) = f(q)/q$; ; see e.g. Theorem VII.4(ii) in [2].

We next turn our attention to composition of functions; here are some classical properties

Proposition 8 *Consider two functions $f, g :]0,\infty[\to [0,\infty[$. Then we have the implications*

$$f, g \in \mathbf{B}_2 \implies f \circ g \in \mathbf{B}_2\,,$$

$$f \in \mathbf{CM} \text{ and } g \in \mathbf{B}_2 \implies f \circ g \in \mathbf{CM}\,,$$

$$f \in \mathbf{B}_1 \text{ and } g \in \mathbf{B}_2 \implies f \circ g \in \mathbf{B}_1\,.$$

The first statement in Proposition 8 is related to the celebrated subordination of Bochner (see, e.g. Section 3.9 in [6] or Chapter 6 in [9]); more precisely if σ and τ are two independent subordinators with respective Bernstein functions f_σ and f_τ, then $\sigma \circ \tau$ is again a subordinator whose Bernstein function is $f_\tau \circ f_\sigma$. The second statement is a classical result which can be found as Criterion 2 on page 441 in Feller [5]; it is also related to Bochner's subordination.

Finally we turn our attention to inverses.

Proposition 9 *Consider a function* $f :]0,\infty[\to]0,\infty[$. *Then*

$$f \in \mathbf{B}_2 \cup \mathbf{B}_3 \implies 1/f \in \mathbf{CM}\,.$$

Further, if f^{-1} *denotes the inverse of* f *when the latter is a bijection, then*

$$f \in \mathbf{B}_3\,, f \not\equiv 0 \implies f^{-1} \in \mathbf{B}_2\,.$$

We mention that if $f \in \mathbf{B}_3$, the completely monotone function $1/f$ is the Laplace transform of the so-called scale function of the Lévy process X with no positive jumps which has Laplace exponent f. See Theorem VII.8 in [2]. On the other hand, f^{-1} is the Bernstein function of the subordinator of first-passage times $T_t := \inf\{s \geq 0 : X_s > t\}$; see e.g. Theorem VII.1 in [2]. Finally, in the case when $f \in \mathbf{B}_2$ is a Bernstein function, the completely monotone function $1/f$ is the Laplace transform of the renewal measure $U(dx) = \int_0^\infty \mathbb{P}(\sigma_t \in dx)dt$, where σ is a subordinator with Bernstein function f.

Corollary 10 *Let* $\Psi \not\equiv 0$ *be a function in* $\mathbf{B}_3$, *and denote by* $\Phi = \Psi^{-1} \in \mathbf{B}_2$ *its inverse bijection. Then* $q \to 1/\Phi'(q)$ *and* $\mathrm{Id}/\Phi : q \to q/\Phi(q)$ *are Bernstein functions. Furthermore* $\frac{1}{\Phi\Phi'} : q \to 1/(\Phi(q)\Phi'(q))$ *is completely monotone.*

Proof: We know from Propositions 6 and 9 that both Φ and Ψ' are Bernstein functions. We conclude from Proposition 8 that $1/\Phi' = \Psi' \circ \Phi$ is again in $\mathbf{B}_2$.

Similarly, we know from Proposition 7 that $q \to \Psi(q)/q$ is a Bernstein function, and composition on the right by the Bernstein function Φ yields Id/Φ that is again in $\mathbf{B}_2$.

Finally, we can write $1/(\Phi\Phi') = f \circ \Phi$ where $f(q) = \Psi'(q)/q$. We know from Proposition 6 that $\Psi' \in \mathbf{B}_2$, so $f \in \mathrm{CM}$ by Proposition 7. Since $\Phi \in \mathbf{B}_2$, we conclude from Proposition 8 that $f \circ \Phi \in \mathbf{CM}$. ∎

If $\Phi = \Psi^{-1}$ is the Bernstein function given by the inverse of a function $\Psi \in \mathbf{B}_3$, the Bernstein function $1/\Phi'$ is the exponent of the subordinator L^{-1} defined as the inverse of the local time at 0 of the Lévy process with no positive jumps and Laplace exponent Ψ. See e.g. Exercise VII.2 in [2]. On the other hand, Id/Φ is then the Bernstein function of the decreasing ladder times, see Theorem VII.4(ii) in [2]. The interested reader is also referred to [1] for further factorizations for Bernstein functions which arise naturally for Lévy processes with no positive jumps, and their probabilistic interpretations.

Next, recall that a function $f :]0,\infty[\to \mathbb{R}_+$ is called a Stieltjes transform if it can be expressed in the form

$$f(q) = b + \int_{[0,\infty[} \frac{\nu(dt)}{t+q}\,, \qquad q > 0\,,$$

where $b \geq 0$ and ν is a Radon measure on $\mathbb{R}_+$ such that $\int_{[0,\infty[}(1 \wedge t^{-1})\nu(dt) < \infty$. Equivalently, a Stieltjes transform is the Laplace transform of a Radon measure μ on $\mathbb{R}_+$ of the type $\mu(dx) = b\delta_0(dx) + h(x)dx$, where $b \geq 0$ and h is a completely monotone function which belongs to $L^1(\mathrm{e}^{-qx}dx)$ for every $q > 0$; see e.g. Section 3.8 in [6].

Corollary 11 *Let $f \in \mathbf{B}_2$ be a Bernstein function such that its derivative f' is a Stieltjes transform. Then for every Bernstein function $g \in B_2$, the function $f \circ \frac{1}{g}$ is completely monotone.*

Proof: We can write

$$f(q) \;=\; a + bq + \int_0^q dr \int_0^\infty dx \mathrm{e}^{-rx} h(x)\,, \qquad q > 0\,,$$

where $a, b \geq 0$ and $h \in \mathbf{B}_1$. Thus

$$f(q) \;=\; a + bq + \int_0^\infty dx (1 - \mathrm{e}^{-qx}) \frac{h(x)}{x}\,, \qquad q > 0\,,$$

and then

$$f \circ \frac{1}{g}(q) \;=\; a + \frac{b}{g(q)} + \int_0^\infty dx (1 - \mathrm{e}^{-x/g(q)}) \frac{h(x)}{x}\,.$$

We already know from Proposition 9 that $a + b/g \in \mathbf{CM}$. The change of variable $y = x/g(q)$ yields

$$\int_0^\infty dx (1 - \mathrm{e}^{-x/g(q)}) \frac{h(x)}{x} \;=\; \int_0^\infty (1 - \mathrm{e}^{-y}) h(yg(q)) \frac{dy}{y}\,.$$

For each fixed $y > 0$, yg is a Bernstein function, so by Proposition 8, the function $q \to h(yg(q))$ is completely monotone.

We conclude that for every integer $n \geq 0$,

$$(-1)^n \frac{\partial^n}{\partial q^n} (f \circ \frac{1}{g})(q) \;=\; \int_0^\infty (-1)^n \frac{\partial^n}{\partial q^n} (h(yg(\cdot))(q)(1 - \mathrm{e}^{-y}) \frac{dy}{y} \;\geq\; 0\,,$$

which establishes our claim. ■

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